K19P 0172

Reg. No. : P B7PS MM1316

Name : Yamuna

IV Semester M.Sc. Degree (Reg.) Examination, April 2019
(2017 Admission Onwards)
MATHEMATICS
MAT4C15 : Operator Theory

Time : 3 Hours | Max. Marks : 80

PART – A

Answer **four** questions from this Part. **Each** question carries 4 marks.

1. Let X be a normed space over K. If $A, B \in BL(X)$ and $k \neq 0$, then prove that $k \in \sigma(AB)$ if and only if $k \in \sigma(BA)$.

2. $x_n \to^w x$ and $y_n \to^w y$ in a normed space X then show that $x_n + y_n \to^w x + y$.

3. Interpret uniform convexity geometrically.

4. Define numerical range of an operator on a Hilbert space and prove or disprove that it is closed subset of K.

5. Let E be a measurable subset of $\mathbb{R}$ and $H = L^2(E)$. Fix z in $L^\infty(E)$ and define $A(x) = zx$, $x \in H$. Show that A is normal.

6. Let $u_1, u_2, \ldots$ constitute an orthonormal basis for H. Suppose that $A \in BL(H)$ is defined by a matrix M with respect to $u_1, u_2, \ldots$. Assume that M is triangular. Then show that A is normal if and only if M is diagonal. **(4×4=16)**

PART – B

Answer **four** questions from this Part without omitting **any** Unit. **Each** question carries **16** marks.

Unit – I

7. a) Let X be a normed space and $A \in BL(X)$ be of finite rank. Then show that $\sigma_e(A) = \sigma_a(A) = \sigma(A)$.

   b) Let X a Banach space. If $A, B \in BL(X)$, A is invertible and $\epsilon = \|(A-B)A^{-1}\| < 1$, then show that B is invertible, $B^{-1} = A^{-1}\sum_{n=0}^{\infty} \left[(A-B)A^{-1}\right]^n$, $\|B^{-1}\| \leq \frac{\|A^{-1}\|}{1-\epsilon}$ and $\|B^{-1} - A^{-1}\| \leq \frac{\|A^{-1}\|\epsilon}{1-\epsilon}$.

P.T.O.

8. a) State and prove Spectral radius formula.
   b) Let X be a normed space. Then prove that if X' is separable, so is X.

9. a) Show that the dual of c_0 with the norm $\|\cdot\|_\infty$ is linearly isometric to l^1.
   b) Let X be a normed space and $\{x_n\}$ be a sequence in X. Then prove that $\{x_n\}$ is weak convergent in X if and only if
      i) (x_n) is a bounded sequence in X and
      ii) there is some $x \in X$ such that $x'(x_n) \to x'(x)$ for every x' in some subset of X' whose span is dense in X'.

Unit – II

10. a) Let X be a Banach space which is uniformly convex in some equivalent norm. Then prove that X is reflexive.
    b) Define compact linear map and give an example.

11. a) Let X and Y be normed spaces and $F : \in BL(X, Y)$. If $F \in CL(X, Y)$, then prove that $F' \in CL(Y'.X')$. Also prove the converse if Y is a Banach space.
    b) Let X be normed space and $A \in CL(X)$, and $0 \neq k \in K$. If (x_n) is a bounded sequence in X such that $A(x_n) - kx_n \to y$ in X, then prove that there is a subsequence (x_{n_j}) of (x_n) such that $x_{nj} \to x$ in X and $A(x) - kx = y$.

12. a) Let X be a linear space, $A : X \to X$ linear and $A(x_n) = k_n x_n$ for some $0 \neq x_n \in X$ and $k_n \in K$, n = 1, 2.... Let $k_n \neq k_m$ whenever $n \neq m$. Then prove that $\{x_1, x_2, ...\}$ is linearly independent subset of X.
    b) Let X be a normed space and $A \in CL(X)$. Then prove that every eigenspace of A corresponding to a nonzero eigenvalue of A is finite dimensional.

Unit – III

13. a) Define invertible operator. Also give an example of an invertible operator.
    b) Let H be a Hilbert space. Consider $A \in BL(H)$. Then prove that $Z(A) = R(A^*)^\perp$ and $Z(A^*) = R(A)^\perp$. Also prove that A is injective if and only if $R(A^*)$ is dense in H, and A^* is injective if and only if R(A) is dense in H.
    c) Define self-adjoint operator and give an example.

14. a) Let H be a Hilbert space. Consider $A \in BL(H)$ and A be self adjoint. Then prove that $\|A\| = \sup\{|\langle A(x),x\rangle| : x \in H, \|x\| \leq 1\}$.

b) Let H be a Hilbert space and (A_n) be a sequence in BL(H) and $A \in BL(H)$ be such that $\|A_n - A\| \to 0$ as $n \to \infty$. If each A_n is self adjoint unitary or normal, then prove that A is self adjoint, unitary or normal respectively.

15. a) Let H be a Hilbert space and $A \in BL(H)$. Then prove that $\sigma_e(A) \subset \sigma_a(A)$ and $\sigma(A) = \sigma_a(A) \cup \left\{k : \bar{k} \in \sigma_e(A^*)\right\}$.

b) Let H be a finite dimensional Hilbert space over K and $A \in BL(H)$. Suppose that there is an orthonormal basis for H consisting of eigen values of A. Then prove that A is a normal operator. If $K = \mathbb{R}$, then prove that A is in fact a self adjoint operator.

(4×16=64)

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Reg. No. : B7PSMM 1316

Name : Yamuna

IV Semester M.Sc. Degree (Reg.) Examination, April 2019
(2017 Admission Onwards)
MATHEMATICS
MAT4C16 : Differential Geometry

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any four** questions. **Each** question carries **4** marks.

1. Define a vector field and illustrate it with an example.

2. Let $f : U \to \mathbb{R}$ be a smooth function on U, U open in $\mathbb{R}^n$. Show that the graph of f is an n-surface in $\mathbb{R}^{n+1}$.

3. Show that the spherical image of an n-surface S with orientation $\mathbb{N}$ is the reflection through the origin of the spherical image of S with orientation $-\mathbb{N}$.

4. Find the velocity, the acceleration and the speed of the parametrized curve $\alpha(t) = (\cos t, \sin t, t)$.

5. Define length of a parametrized curve in $\mathbb{R}^{n+1}$ and show that it is invariant under reparametrization.

6. Describe a parametrized torus in $\mathbb{R}^4$. **(4×4=16)**

PART – B

Answer **any four** questions without omitting any Unit. **Each** question carries **16** marks.

Unit – I

7. a) Let $\mathbb{X}$ be a smooth vector field on an open set $U \subset \mathbb{R}^{n+1}$ and let $p \in U$. Prove the existence of the maximal integral curve of $\mathbb{X}$ through p.

   b) Sketch typical level curves and the graph of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x_1, x_2) = -x_1^2 + x_2^2$.

P.T.O.

8. a) Let U be an open set in $\mathbb{R}^{n+1}$ and let $f : U \to \mathbb{R}$ be smooth. Let $p \in U$ be a regular point of f and let $c = f(p)$. Prove that the set of all vectors tangent to $f^{-1}(c)$ at p is equal to $[\nabla f(p)]^{\perp}$.

b) Let $f : U \to \mathbb{R}$ be a smooth function and let $\alpha : I \to U$ be an integral curve of ∇f.

i) Show that $\frac{d}{dt}(f \circ \alpha)(t) = \| \nabla f(\alpha(t)) \|^2$ for all $t \in I$.

ii) Show that for each $t_0 \in I$, the function f is increasing faster along α at $\alpha(t_0)$ then along any other curve passing through $\alpha(t_0)$ with the same speed.

9. a) State and prove the Lagrange multiplier theorem.

b) Prove that each connected n-surface in $\mathbb{R}^{n+1}$ has exactly two orientations.

c) Define an oriented n-surface. Give an example of an "unoriented 2-surface" with justification.

Unit – II

10. a) Prove that for a compact connected oriented n-surface S in $\mathbb{R}^{n+1}$ with $S = f^{-1}(c)$, $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is a smooth function with $\nabla f(p) \neq 0$ for all $p \in S$, the Gauss map $N : S \to S^n$ is onto.

b) Prove that geodesics have constant speed.

11. a) Let S be an n-surface in $\mathbb{R}^{n+1}$, let $\alpha : I \to S$ be a parametrized curve in S, let $t_0 \in I$ and let $v \in S_{\alpha(t_0)}$. Prove that there exists a unique vector field $\mathbb{V}$ tangent to S along α, which is parallel and has $\mathbb{V}(t_0) = v$.

b) Let S be an n-surface in $\mathbb{R}^{n+1}$, let $\alpha : I \to S$ be a parametrized curve and let $\mathbb{X}$ and $\mathbb{Y}$ be vector fields tangent to S along α. Verify that

i) $(\mathbb{X} + \mathbb{Y})' = \mathbb{X}' + \mathbb{Y}'$ and

ii) $(f\,\mathbb{X})' = f'\,\mathbb{X} + f\,\mathbb{X}'$

for all smooth function f along α.

12. a) Prove that the Weingarten map is self-adjoint.

b) Define a local parametrization of plane curve. Find a global parametrization of the curve oriented by $\nabla f / \| \nabla f \|$ where f is the function defined by the left side of the equation $ax_1 + bx_2 = c$, $(a, b) \neq (0, 0)$.

Unit – III

13. a) On each compact oriented n-surface S in $\mathbb{R}^{n+1}$, prove that there exists a point p such that the second fundamental form at p is definite.

b) Define a differential 1-form. Prove that for each 1-form W on U(U open in $\mathbb{R}^{n+1}$) there exist unique functions $f_i : U \to \mathbb{R}$, $i = 1, 2, ..., n+1$ such that $W = \sum_{i=1}^{n+1} f_i \, dx_i$.

14. a) Find the Gaussian curvature of the ellipsoid $(x_1^2 / a^2) + (x_2^2 / b^2) + (x_3^2 / c^2) = 1$ (a, b, c all $\neq 0$) oriented by its outward normal.

b) Let ψ be the parametrized torus in $\mathbb{R}^3$:

$\psi(\theta, \phi) = ((a + b \cos \phi) \cos\theta, (a + b \cos\phi) \sin\theta, b \sin \phi)$

Find its Gaussian curvature.

15. a) Define an n-surface S in $\mathbb{R}^{n+k}$ $(k \geq 1)$. With usual notations express S in the form $S = \bigcap_{i=1}^{k} f_i^{-1}(c_i)$. Define the tangent space S_p at $p \in S$ and the normal space to S at p. Illustrate a 1-surface in $\mathbb{R}^3$ with its tangent space and normal space at a point p.

b) State and prove the inverse function theorem for n-surfaces. (4×16=64)

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K19P 0174

Reg. No. : B7PSMM1316

Name : Yamuna

IV Semester M.Sc. Degree (Reg.) Examination, April 2019
(2017 Admission Onwards)
MATHEMATICS
MAT 4E01 : Commutative Algebra

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any four** questions. **Each** question carries **4** marks.

1. Let A be a ring in which every element x satisfies $x^n = x$ for some $n > 1$ (depending on x). Show that every prime ideal in A is maximal.

2. Let P be a prime ideal of a ring A. Prove that P[x] is a prime ideal in A[x].

3. Let S be a multiplicatively closed subset of a ring A and let M be a finitely generated A-module. Prove that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$.

4. Let I be an ideal of a ring A. If $I = r(I)$, show that I has no embedded prime ideals, where r(I) denotes radical of I.

5. Let $A \subseteq B$ be rings, B integral over A. If $x \in A$ is a unit in B, prove that B is a unit in A.

6. If A[x] is Noetherian, is A necessarily Noetherian ? Justify your answer. **(4×4=16)**

P.T.O.

PART – B

Answer **any four** questions without omitting **any** Unit. **Each** question carries **16** marks.

Unit – I

7. a) Let A be a ring $\neq 0$. Prove that A is a field if and only if every homomorphism of A into a nonzero ring B is injective.

   b) Prove that the nilradical of a ring A is the intersection of all prime ideals of A.

8. a) Let I and J be coprime ideals of a ring A. Prove that (i) $IJ = I \cap J$ (ii) There is a natural isomorphism of the ring $A/(I \cap J)$ onto the ring $(A/I) \times (A/J)$.

   b) Prove that M is a finitely generated A-module if and only if M is isomorphic to a quotient of A^n for some integer $n > 0$.

9. a) If N and P are submodules of an A-module M, prove that $Ann((N + P)/N) = (N : P)$.

   b) State and prove Nakayama's lemma.

Unit – II

10. a) Let $g : A \to B$ be a ring homomorphism such that g(s) is a unit in B for all $s \in S$ where S is a multiplicatively closed subset of A. Prove that there is a unique ring homomorphism $h : S^{-1} A \to B$ such that $g = h \circ f$ where $f : A \to S^{-1} A$ is the ring homomorphism defined by $f(x) = x/1$.

    b) Prove that the ring $S^{-1} A$ and the homomorphism $f : A \to S^{-1} A$ have the following properties :

    i) $s \in S$ is a unit $\Rightarrow$ f(s) is a unit in $S^{-1} A$;

    ii) $f(a) = 0 \Rightarrow as = 0$ for some $s \in S$;

    iii) Every element of $S^{-1} A$ is of the form $f(a) f(s)^{-1}$ for some $a \in A$ and some $s \in S$.

    Conversely prove that these three conditions determine the ring $S^{-1} A$ upto isomorphism.

11. a) Let M be an A-module and S a multiplicatively closed subset of A. Prove that the $S^{-1} A$ modules $S^{-1} M$ and $S^{-1} A \otimes_A M$ are isomorphic.

    b) Let M be an A-module. Prove that M = 0 if and only if $M_m = 0$ for all maximal ideals m of A.

12. a) State and prove first uniqueness theorem.

b) Prove that the isolated primary components of a decomposable ideal I are uniquely determined by I.

Unit – III

13. a) Let $A \subseteq B$ be integral domains, B integral over A. Prove that B is a field if and only if A is a field.

b) State and prove the going down theorem.

14. a) Prove that M is a Noetherian A-module if and only if every submodule of M is finitely generated.

b) Suppose that M has a composition series of length n. Then prove that every composition series of M has length n and every chain in M can be extended to a composition series.

15. a) If A is Noetherian, then prove that the polynomial ring A[x] is Noetherian.

b) In a Noetherian ring A, prove that every ideal has a primary decomposition.

(4×16=64)