K18P 1032

Reg. No. : ....

Name : ....

Third Semester M.Sc. Degree (Reg.) Examination, October 2018
MATHEMATICS
(2017 Admn. Onwards)
MAT3C11 : Number Theory

Time : 3 Hours Max. Marks: 80

PART – A

Answer **any four** questions. **Each** question carries **4** marks.

1. Prove that every number of the form $2^{a-1}(2^a-1)$ is perfect if $2^a - 1$ is prime.
2. Solve the congruence $5x \equiv 3 \pmod{24}$.
3. If p is an odd prime, prove that $\sum_{r=1}^{p-1} r(r|p) = 0$, if $P \equiv 1 \pmod 4$.
4. If $m \geq 1$, $(a, m) = 1$ and $f = \exp_m(a)$, then prove that $a^k \equiv a^h \pmod m$ if and only if $k \equiv h \pmod f$.
5. Let $\mathbb{Z}$ be a $\mathbb{Z}$– module with the obvious action. Find all the submodules.
6. Let $K = Q(\zeta)$, where $\zeta = e^{2\pi i/p}$ for a rational prime p. In the ring of integers of $\mathbb{Z}[\zeta]$, show that $\alpha \in \mathbb{Z}[\zeta]$ is a unit if and only if $N_K(\alpha) = \pm 1$. **(4×4=16)**

PART – B

Answer **any four** questions without omitting any Unit. **Each** question carries **16** marks.

Unit – I

7. a) State and prove the fundamental theorem of arithmetic.

   b) Define the Euler totient function $\varphi(n)$ and derive a product formula for it.

P.T.O.

8. a) Define the Dirichlet product $f * g$ of two arithmetic functions. If both g and $f * g$ are multiplicative, prove that f is also multiplicative.

   b) Let f be multiplicative. Prove that f is completely multiplicative if and only if $f^{-1}(n) = \mu(n) f(n)$ for all $n \geq 1$.

   c) Prove that $\varphi^{-1}(n) = \sum_{d|n} d\mu(d)$.

9. a) State and prove Lagrange's theorem on polynomial congruences.

   b) State the principle of cross classification. Given integers r, d and k such that $d|k$, $d > 0$, $k \geq 1$ and $(r, d) = 1$. Then prove that the number of elements of the set $S = \{r + td : t = 1, 2, ..., k/d\}$ which are relatively prime to k is $\varphi(k)/\varphi(d)$.

Unit – II

10. a) State and prove the quadratic reciprocity law.

    b) Determine whether 219 is a quadratic residue or non-residue modulo 383.

11. a) Let p be an odd prime and let d be any positive divisor of $p - 1$. Prove that in every reduced residue system modulo p there are $\varphi(d)$ numbers a such that $\exp_p(a) = d$.

    b) If $\alpha \geq 3$, prove that there are no primitive roots mod 2^α.

12. a) Encipher the message HAVEANICETRIP using a Vigenere cipher with the keyword MATH.

    b) The ciphertext ALXWU VADCOJO has been enciphered with the cipher

    $C_1 \equiv 4P_1 + 11P_2 \pmod{26}$, $C_2 \equiv 3P_1 + 8P_2 \pmod{26}$. Derive the plain text.

    c) Find the unique solution of the knapsack problem

    $51 = 3x_1 + 5x_2 + 9x_3 + 18x_4 + 37x_5$.

Unit – III

13. a) Let G be a free abelian group of rank n with basis $\{x_1, \ldots, x_n\}$. Suppose (a_{ij}) is an $n \times n$ matrix with integer entries. Prove that the elements

$$y_i = \sum_{j=1}^{n} a_{ij} x_j, \ (i = 1, \ldots, n)$$ form a basis of G if and only if (a_{ij}) is unimodular.

b) Prove that every subgroup H of a free abelian group of rank n is free of rank $s \leq n$.

14. a) If K is a number field then prove that $K = Q(\theta)$ for some algebraic number θ.

b) Prove that a complex number θ is an algebraic integer if and only if the additive group generated by all powers $1, \theta, \theta^2, \ldots$, is finitely generated.

15. a) Prove that the ring of integers of the cyclotomic field $Q(\zeta)$, where $\zeta = e^{2\pi i/p}$, p an odd prime is $\mathbb{Z}[\zeta]$.

b) Prove that the discriminant of $Q(\zeta)$, where $\zeta = e^{2\pi i/p}$, p an odd prime is $(-1)^{(p-1)/2} p^{p-2}$.

(4×16=64)

K18P 1033

Reg. No. : ....

Name : ....

Third Semester M.Sc. Degree (Reg.) Examination, October 2018
MATHEMATICS
(2017 Admn. Onwards)
MAT3C12 : Functional Analysis

Time : 3 Hours Max. Marks : 80

PART – A

Answer **four** questions from this Part. **Each** question carries **4** marks.

1. Show that if $x_n \to x$ in l^1 then $x_n \to x$ in l^2.

2. Give an example of an element in $L^1(\mathbb{R})$ but not in $L^2(\mathbb{R})$ and prove your claim.

3. Show that the norms $||.||_1$ and $||.||_2$ on K^n, n = 1, 2, . . . are equivalent.

4. If X is an infinite dimensional space then prove that it contains a hyperspace which is not closed.

5. Let X be a normed linear space and (x_n) be a sequence in X. Prove or disprove : (x_n) converges in X if and only if $f(x_n)$ converges in K for every $f \in X'$.

6. Give an example of a function on $K^n \times K^4$ which is linear in the first variable and conjugate symmetric but not an inner product. Also prove your claim.

PART – B

Answer **4** questions from this Part without omitting **any** Unit. **Each** question carries **16** marks.

Unit – I

7. a) State and prove Jenson's inequality.
   b) State and prove Riesz Lemma.

P.T.O.

8. a) Show that a linear map F from a normed space X to a normed space Y is a homeomorphism if and only if there are $\alpha, \beta > 0$ such that $\beta\|x\| \leq \|F(x)\| \leq \alpha\|x\|$ for all $x \in X$. In case there is a linear homeomorphism from X onto Y then prove that X is complete if and only if Y is complete.

   b) Let X denote a subspace of B(T) with the sup norm, $1 \in X$ and f be a linear functional on X. If f is continuous and $\|f\| = f(1)$, then prove that f is positive. Conversely, if $\text{Re}\,x \in X$ whenever $x \in X$ and if f is positive, then prove that f is continuous and $\|f\| = f(1)$.

9. a) Let X and Y be normed spaces and $X \neq \{0\}$. Then prove that BL(X, Y) is a Banach space in the operator norm if and only if Y is a Banach space.

   b) Let X be a normed space and Y be a Banach space. Let X_0 be a dense subspace of X and $F_0 \in BL(X_0, Y)$. Then prove that there is a unique $F \in BL(X, Y)$ such that $F|X_0 = F_0$ and $\|F\| = \|F_0\|$.

Unit – II

10. a) Let X be a normed space and E be a subset of X. Then prove that E is bounded in X if and only if f(E) is bounded in K for every $f \in X'$.

    b) Define closed map. If a closed map F is bijective then prove that its inverse F^{-1} is also a closed map.

11. a) State and prove closed graph theorem.

    b) Define open map and give an example.

12. a) Let X and Y be normed spaces and $F : X \to Y$ be linear. Then prove that F is an open map if and only if there exists some $\gamma > 0$ such that for every $y \in Y$, there is some $x \in X$ with $F(x) = y$ and $\|x\| \leq \gamma\|y\|$.

    b) Show that the open mapping theorem may not hold if the range of the linear map is not a Banach space.

Unit – III

13. a) State and prove parallelogram law.

    b) Let u_α be an orthonormal set in a Hilbert space H. Then prove that the following conditions are equivalent.

       i) $\{u_\alpha\}$ is an orthonormal basis for H.

       ii) For every $x \in H$, we have

       $x = \Sigma_n \langle x, u_n \rangle u_n$, where $\{u_1, u_2, \ldots\} = \{u_\alpha : \langle x, u_n \rangle u_n u_n$, where

       $\{u_1, u_2, \ldots\} = \{u_\alpha : \langle x, u_\alpha \rangle \neq 0\}$.

iii) For every $x \in H$, we have $\|x\|^2 = \Sigma_n |\langle x, u_n \rangle|^2$, where $\{u_1, u_2, \ldots\} = \{u_\alpha : \langle x, u_\alpha \rangle \neq 0\}$.

iv) Span $\{u_\alpha\}$ is dense in H.

v) If $x \in H$ and $\langle x, u_\alpha \rangle = 0$ for all α, then $x = 0$.

14. a) Let X be an inner product space and $f \in X'$. Let $\{u_1, u_2, \ldots\}$ be an orthogonal set in X. Then prove that $\sum_n |f(u_n)|^2 \leq \|f\|^2$.

b) Prove that the projection theorem does not hold for an incomplete inner product space.

15. a) Let (x_n) be a bounded sequence in a Hilbert space H then prove that it has a weak convergent subsequence.

b) Let H be a Hilbert space over K. If F_1 and F_2 are closed subspaces of H, then prove that $(F_1 + F_2)^\perp$ equals the closure of $F_1^\perp + F_2^\perp$.

K18P 1034

Reg. No. : ....

Name : ....

Third Semester M.Sc. Degree (Reg.) Examination, October 2018
MATHEMATICS
(2017 Admn. Onwards)
MAT3C13 : Complex Function Theory

Time : 3 Hours　　　　Max. Marks : 80

PART – A

Answer **any four** questions. **Each** question carries **4** marks.

1. Prove that an elliptic function without poles is a constant.
2. Define the Weierstrass sigma function $\sigma(z)$ and show that it is an odd function.
3. Find a meromorphic function in the plane with a pole at every integer.
4. Suppose that f(z) is analytic in a region G which is symmetric with respect to the real axis and f(x) is real for all x in $G \cap \mathbb{R}$. Prove that $f(z) = \overline{f(\overline{z})}$ for all z in G.
5. If u is harmonic, show that $f = u_x - iu_y$ is analytic.
6. Prove or disprove : every harmonic function is subharmonic. **(4×4=16)**

PART – B

Answer **any four** questions **without** omitting any **unit**. **Each** question carries **16** marks.

Unit – I

7. a) Prove that a discrete module consists either of zero alone, of the integral multiples nw of a single complex number $w \neq 0$, or of all linear combinations $n_1w_1 + n_2w_2$ with integral coefficients of two numbers w_1 and w_2 with nonreal ratio w_2/w_1.
   b) Prove that any two bases of the same period module are connected by a unimodular transformation.

P.T.O.

8. a) Describe the construction of the Weierstrass P-function.

b) Prove that addition theorem for the P-function :

$$P(z+u) = -P(z) - P(u) + \frac{1}{4}\left(\frac{P'(z)-P'(u)}{P(z)-P(u)}\right)^2$$

9. a) Define the Riemann zeta function $\zeta(z)$. Prove that for $\operatorname{Re} z > 1$, $\zeta(z)$

$$\Gamma(z) = \int_0^\infty (e^t - 1)^{-1} t^{z-1}\, dt.$$

b) Derive Riemann's functional equation $\zeta(z) = 2\,(2\pi)^{z-1}\,\Gamma(1-z)\;\zeta(1-z)$

$\sin\left(\frac{1}{2}\pi z\right)$ for $-1 < \operatorname{Re} z < 0$.

Unit – II

10. a) Let K be a compact subset of the region G. Prove that there are straight line segments $r_1, \ldots, r_n$ in G – K such that for every function f in H(G),

$f(z) = \sum_{k=1}^{n} \frac{1}{2\pi i} \int_{r_k} \frac{f(w)}{w-z}\, dw$ for all z in K and the line segments form a finite number of closed polygons.

b) Let G be an open connected subset of $\mathbb{C}$. If $n(r, a) = 0$ for every closed rectifiable curve r in G and every point a in $\mathbb{C} - G$, then prove that $\mathbb{C}_\infty - G$ is connected.

11. a) State and prove Mittag-Leffler's theorem.

b) Define analytic continuation along a path.

12. a) State and prove Schwarz reflection principle.

b) With usual assumptions, what is the meaning of saying that a function element (f, D) admits unrestricted analytic continuation in G ?

c) State monodromy theorem.

Unit – III

13. a) State and prove the mean value theorem for harmonic functions.

    b) Let $D = \{z : |z| < 1\}$ and suppose that $f : \partial D \to \mathbb{R}$ is a continuous function. Prove that there is a unique continuous function $u : D^{-} \to \mathbb{R}$ such that :

    i) $u(z) = f(z)$ for all z in ∂D and

    ii) $u(z)$ is harmonic in D.

14. a) If $u : G \to \mathbb{R}$ is a continuous function which has the mean value property, prove that u is harmonic.

    b) State and prove Harnack's theorem.

15. a) Let G be a region and $f : \partial_{\infty} G \to \mathbb{R}$ a continuous function. Prove that $u(z) = \sup \{\varphi(z) : \varphi \in P(f, G)\}$ defines G harmonic function u on G.

    b) Derive Jensen's formula.

(4×16=64)

K18P 1035

Reg. No. : ....

Name : ....

Third Semester M.Sc. Degree (Reg.) Examination, October 2018
MATHEMATICS
(2017 Admn. Onwards)
MAT3C14 : Advanced Real Analysis

Time : 3 Hours Max. Marks : 80

PART – A

Answer **four** questions from this part. **Each** question carries **4** marks.

1. Give an example of a sequence of functions which converges pointwise but not uniformly.
2. If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, prove that $\{f_n + g_n\}$ converges uniformly on E.
3. Consider $f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2x}$. On what intervals does it fail to converge uniformly ?
4. Show that e^x defined on $\mathbb{R}^1$ satisfy the relation $(e^x)' = e^x$.
5. Define orthogonal system of functions and give an example.
6. Prove that $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$. **(4×4=16)**

PART – B

Answer **4** questions from this part without omitting **any** Unit. **Each** question carries **16** marks.

Unit – I

7. a) If $\{f_n\}$ is a sequence of continuous function on E, and if $f_n \to f$ uniformly on E, then show that f is continuous on E.

   b) If $f_n \in \mathcal{R}(\alpha)$ on [a, b] and if $f(x) = \sum_{n=1}^{\infty} f_n(x)\ (a \le x \le b)$, the series converging uniformly on [a, b], then prove that $\int_a^b f d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n d\alpha$.

P.T.O.

8. a) Even if $\{f_n\}$ is a uniformly bounded sequence of continuous functions on a compact set E, prove that there need not exist a subsequence which converges pointwise on E.

   b) If $\{f_n\}$ is a pointwise bounded sequence of complex functions on a countable set E, then prove that $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(x)\}$ converges for every $x \in E$.

9. State and prove Stone Weierstrass theorem.

Unit – II

10. a) Suppose $\sum c_n$ converges. Put $f(x) = \sum_{n=0}^{\infty} c_n x^n (-1 < x < 1)$. Then prove that $\lim_{x \to 1} f(x) = \sum_{n=0}^{\infty} c_n$.

    b) Define analytic functions and give an example.

11. a) Suppose the series $\sum a_n x^n$ and $\sum b_n x^n$ converge in the segment $S = (-R, R)$. Let E be the set of all $x \in S$ at which $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$. If E has a limit point in S, then prove that $a_n = b_n$ for n = 0, 1, 2, ... Hence $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$ for all $x \in S$.

    b) Let $\{\phi_n\}$ be orthonormal on [a, b]. Let $s_n(x) = \sum_{m=1}^{n} c_m \phi_m(x)$ be the n^{th} partial sum of the Fourier series of f, and suppose $t_n(x) = \sum_{m=1}^{n} \gamma_m \phi_m(x)$. Then prove that $\int_a^b |f - s_n|^2 dx \le \int_a^b |f - t_n|^2 dx$, and equality holds if and only if $\gamma_m = c_m (m = 1, 2, ..., n)$.

12. a) If, for some x, there are constants $\delta > 0$ and $M < \infty$ such that

    $|f(x + t) - f(x)| \le M|t|$ for all $t \in (-\delta, \delta)$, then prove that $\lim_{N \to \infty} S_N(f; x) = f(x)$.

    b) If $f(x) = 0$ for all x in some segment J, then prove that $\lim S_N(f; x) = 0$ for every $x \in J$.

    c) If f is continuous (with period 2π) and if $\epsilon > 0$, then prove there is a trigonometric polynomial P such that $|P(x) - f(x)| < \epsilon$ for all real x.

Unit – III

13. a) Define the dimension of a vector space and give an example.

    b) Define basis of a vector space.

    c) Let r be a positive integer. If a vector space X is spanned by a set of r vectors, then prove that dim $X \leq r$.

14. a) Suppose E is an open set in $\mathbb{R}^n$, f maps E into $\mathbb{R}^m$, f is differentiable at $x_0 \in E$, g maps an open set containing f(E) into $\mathbb{R}^k$, and g is differentiable at $f(x_0)$. Then prove that the mapping F of E into $\mathbb{R}^k$ defined by $F(x) = g(f(x))$ is differentiable at x_0 and $F'(x_0) = g'(f(x_0)f'(x_0))$.

    b) Suppose f maps a convex open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$, f is differentiable in E, and there is a real number M such that $\| f'(x) \| \leq M$ for every $x \in E$. Then prove that $| f(b) - f(a) | \leq M | b - a |$ for all $a \in E$, $b \in E$.

15. State and prove implicit function theorem.

(4×16=64)

K18P 1036

Reg. No. : ....

Name : ....

III Semester M.Sc. Degree (Reg.) Examination, October 2018
MATHEMATICS
(2017 Admn. Onwards)
MAT3E01 : Graph Theory

Time : 3 Hours Max. Marks : 80

Instructions : *1) Answer **any four** questions from Part – **A**. **Each** question carries **4** marks.*

*2) Answer **any 4** questions without omitting **any** Unit from Part – **B**. **Each** question carries **16** marks.*

PART – A

I. Answer **any 4** questions. **Each** question carries **4** marks.

1) Define a (k, l) Ramsay graph and give one example.

2) In a critical graph, prove that no vertex cut is a clique.

3) For a bipartite graph G, show that $\chi'(G) = \Delta$.

4) If G is a planar graph then prove that every subgraph of G is planar.

5) Let l be a flexible vertex labelling of G. If G_l contains a perfect matching M*, then prove that M* is an optimal matching of G.

6) Let u and v be two distinct vertices of a graph G. Then prove that a set S of vertices of G is u – v separating if and only if every u – v path has at least one internal vertex belonging to S.

PART – B

Answer **any 4** questions without omitting any unit. **Each** question carries **16** marks.

Unit – I

II. a) Define the independence number and covering number of a graph and prove that the sum of the independence number and covering number is the number of vertices. **8**

b) Define the Ramsay number r(k, l) and show that $r(k, k) \geq 2^{k/2}$. **8**

P.T.O.

III. a) If a simple graph G contain no K_{m+1}, then prove that G is degree majorised by same complete m-partite graph H. Also show that if G has the same degree sequence as H then $G \cong H$. 8

b) Define the chromatic number χ (G) of a graph G. Give example of a critical graph and a graph which is not critical. Also for a graph G, show that $\chi \leq \Delta + 1$. Give an example of a graph where $\chi = \Delta + 1$. 8

IV. a) For any positive integer k, prove that there exist a k-chromatic graph containing no triangles. 8

b) If G is 4-chromatic, then prove that G contain a subdivision of k_4. 8

Unit – II

V. a) If G is bipartite show that $\chi' = \Delta$. 5

b) Let G be a connected graph that is not an odd cycle, then prove that G has a 2-edge colouring in which both colours are represented at each vertex of degree at least two. 6

c) What is a time tabling problem and explain how one can solve the time tabling problem using edge colouring ? 5

VI. a) Define a dual graph of a graph G and prove or disprove – "Dual of isomorphic plane graph are isomorphic". 6

b) If G is a connected plane graph, then prove that $V - \Sigma + \phi = 2$, further deduce that K_5 is non planar. 10

VII. State and prove Kuratowski's theorem. 16

Unit – III

VIII.a) In a bipartite graph, prove that the number of edges in a maximum matching is equal to the number of vertices in a maximum covering. 10

b) If G is a k-regular bipartite graph with k > 0 then prove that G has a perfect matching. 6

IX. a) Prove that every 3-regular graph without cut edge has a perfect matching. 6

b) Explain in detail the Hungarian method to find an M-augmenting path in a graph and draw its flow-chart. 10

X. a) Let f be a flow on a network N = (V, A) and let f have value d. If $A(X, \bar{X})$ is a cut in N then prove that $d = f(X, \bar{X}) - f(\bar{X}, X)$. Also prove that $d \leq C(X, \bar{X})$. 8

b) State and prove Mengers theorem. 8